Part 6: How to Get Oscillations

We now have obtained enough results to answer the primary goal of our study of the Dynamic System.

How do we get the system to oscillate?

that is

What are the conditions on the parameters of our models to obtain a limit cycle?

Since we are in 2 Dimensions and we have proven the trajectories are bounded, we can apply Poincare-Bendixson. Therefore all trajectories will either converge towards a steady point or to a limit-cycle.

In order to obtain a limit-cycle we need to ensure that all Steady Points are unstable.

Step 1: Choose E and R and Make Sure E is not too large

A necessary condition on E and R=C/D was identified for the steady points to be all unstable

We must have $E < \left(\frac{R}{R+1}\right)^2$

NB: the condition comes from the condition on the trace of the Jacobian for the points associated with the cubic P(U).

If $\frac{R}{R+1} > E \geq \left(\frac{R}{R+1}\right)^2$ the steady point (s) yielded by the cubic are stable

If $1 > E \geq \frac{R}{R+1}$ the steady point S_2 is stable, while for E>1 it is S_1 that is stable.

NB 2: it is unclear whether we can find an adequate combination of parameters (B, B_0, C, D) for every value of E in this range – in particular for E close to the upper bound the condition on the trace may not be met.

Step 2: Select Ad-hoc values for (B, B_0, C, D)

Case 1: R≤1

If you have chosen a value of R less than 1, then the cubic P(U) will only yield one extra steady-point. To find a combination for which the system oscillates, you now need to fulfill two steps

Step 2.1: For R and E fixed find a combination of B and B_0 such as $1 - B_0 > E(1+U)^2$.

This can be done in many ways . Since the condition was not fully studied, I cannot tell if there is a simple equivalent condition on B and B_0.
However, I can recommend a simple numerical approach. The basis for it lies on a simple observation : if we choose B_0=0 then all the positive values of B are possible.
Therefore we can afford to

1) pick up B first
2) for this value of B estimate the largest value of B_0 such as

$$1 - B_0 > E(1+U)^2$$

3) Choose B_0 below this limit value

Step 2.2: Find Adequate Values of C and D

Finding correct values of D can D is straightforward:
1) Calculate $D_{\lim}$
Once B ,B_0, R and E are chosen, compute the corresponding coordinates of the steady point (U,V) and then compute $D_{\lim} = \frac{RU}{V(B_0+U)}\left(\frac{1-B_0}{(1+U)^2} - E\right)$

2) Choose $D<D_{\lim}$ (do not forget to set up $C = R \bullet D$)

Case 2: R>1

If you have chosen a value of R more than 1, then the cubic P(U) will yield between one and three extra steady-points. We recommend you have the case with two roots as it yields one steady point that is not really unstable (0 as eigenvalue).
To find a combination for which the system oscillates, you need to fulfill these two steps

Step 2.1: For R and E fixed find a combination of B and B_0 such as $1 - B_0 > E(1+U)^2$.

Because we are not sure anymore to have one positive root only, this step is more complicated than before. Indeed if we have 3 positive roots for P, we now need to ensure the condition is met for the smallest and largest roots.
However, the simple numerical approach of fixing B is still relevant.

1) pick up B first
2) for this value of B estimate the largest value of B_0 such as
$1 - B_0 > E(1+U)^2$ for all relevant roots
3) Choose B_0 below this limit value

Step 2.2: Find Adequate Values of C and D

Finding correct values of D can D is again straightforward:

1) Calculate D_{lim}

Once B ,B_0, R and E are chosen, compute the corresponding coordinates of the steady point (U,V) and then compute $D_{\mathrm{lim}} = \frac{RU}{V(B_0+U)}\left(\frac{1-B_0}{(1+U)^2} - E\right)$

If you have 3 positive roots for P then you need to calculate D_{lim} for the smallest and largest roots and use the smaller value of D_{lim}

2) Choose $D<D_{lim}$ (do not forget to set up $C = R \bullet D$)

Step 3: Simulate with the selected values (B, B_0, C, D, E).

It will oscillate!!